The Online Min-Sum Set Cover Problem

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Abstract

We consider the online Min-Sum Set Cover (MSSC), a natural and intriguing generalization of the classical list update problem. In Online MSSC, the algorithm maintains a permutation on n elements based on subsets S_1, S_2, \ldots arriving online. The algorithm serves each set S_t upon arrival, using its current permutation π_t , incurring an access cost equal to the position of the first element of S_t in π_t . Then, the algorithm may update its permutation to π_{t+1} , incurring a moving cost equal to the Kendall tau distance of π_t to π_{t+1} . The objective is to minimize the total access and moving cost for serving the entire sequence. We consider the *r*-uniform version, where each S_t has cardinality *r*. List update is the special case where r = 1.

We obtain tight bounds on the competitive ratio of deterministic online algorithms for MSSC against a static adversary, that serves the entire sequence by a single permutation. First, we show a lower bound of $(r+1)(1-\frac{r}{n+1})$ on the competitive ratio. Then, we consider several natural generalizations of successful list update algorithms and show that they fail to achieve any interesting competitive guarantee. On the positive side, we obtain a O(r)-competitive deterministic algorithm using ideas from online learning and the multiplicative weight updates (MWU) algorithm.

Furthermore, we consider efficient algorithms. We propose a memoryless online algorithm, called *Move-All-Equally*, which is inspired by the Double Coverage algorithm for the k-server problem. We show that its competitive ratio is $\Omega(r^2)$ and $2^{O(\sqrt{\log n \cdot \log r})}$, and conjecture that it is f(r)-competitive. We also compare Move-All-Equally against the dynamic optimal solution and obtain (almost) tight bounds by showing that it is $\Omega(r\sqrt{n})$ and $O(r^{3/2}\sqrt{n})$ -competitive.

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1 Introduction

In Min-Sum Set Cover (MSSC), we are given a universe U on n elements and a collection of subsets $S = \{S_1, \ldots, S_m\}$, with $S_t \subseteq U$, and the task is to construct a permutation (or list) π of elements of U. The cost $\pi(S_t)$ of covering a set S_t (a.k.a. the cover time of S_t) with a permutation π is the position of the first element of S_t in π , i.e., $\pi(S_t) = \min\{i \mid \pi(i) \in S_t\}$. The goal is to minimize the overall cost $\sum_t \pi(S_t)$ of covering all subsets of S.

The MSSC problem generalizes various NP-hard problems such as Min-Sum Vertex Cover and Min-Sum Coloring and it is well-studied. Feige, Lovasz and Tetali [25] showed that the greedy algorithm, which picks in each position the element that covers the most uncovered sets, is a 4-approximation (this was also implicit in [11]) and that no $(4-\epsilon)$ -approximation is possible, unless P = NP. Several generalizations have been considered over the years with applications in various areas (we discuss some of those problems and results in Section 1.2).

Online Min-Sum Set Cover. In this paper, we study the online version of Min-Sum Set Cover. Here, the sets arrive online; at time step t, the set S_t is revealed. An online algorithm is charged the *access cost* of its current permutation $\pi_t(S_t)$; then, it is allowed to change its permutation to π_{t+1} at a *moving cost* equal to the number of inversions between π_t and π_{t+1} , known as the Kendall tau distance $d_{\mathrm{KT}}(\pi_t, \pi_{t+1})$. The goal is to minimize the total cost, i.e., $\sum_t (\pi_t(S_t) + d_{\mathrm{KT}}(\pi_t, \pi_{t+1}))$. This is a significant generalization of the classic list update problem, which corresponds to the special case where $|S_t| = 1$ for all sets $S_t \in \mathcal{S}$.

Motivation. Consider a web search engine, such as Google. Each query asked might have many different meanings depending on the user. For example, the query "Python" might refer to an animal, a programming language or a movie. Given the pages related to "Python", a goal of the search engine algorithm is to rank them such that for each user, the pages of interest appear as high as possible in the ranking (see e.g., [23]). Similarly, news streams include articles covering different reader interests each. We want to rank the articles so that every reader finds an article of interest as high as possible. The MSSC problem serves as a theoretical model for practical problems of this type, where we want to aggregate disjunctive binary preferences (expressed by the input sets) into a total order. E.g., for a news stream, the universe U corresponds to the available articles and the sets S_t correspond to different user types. The cost of a ranking (i.e., permutation on U) for a user type is the location of the first article of interest. Clearly, in such applications, users arrive online and the algorithm might need to re-rank the stream (i.e., change the permutation) based on user preferences.

Benchmarks. For the most part, we evaluate the performance of online algorithms by comparing their cost against the cost of an optimal offline solution that knows the input in advance and chooses an optimal permutation π . Note that this solution is *static*, in the sense that it does not change permutations over time. This type of analysis, called *static optimality*, is typical in online optimization and online learning. It was initiated in the context of adaptive data structures by the landmark result of Sleator and Tarjan [44], who showed that *splay trees* are asymptotically as fast as any *static* tree. Since then, it has been an established benchmark for various problems in this area (see e.g. [13, 30]); it is also a standard benchmark for several other problems in online optimization (e.g., online facility location [26, 37], minimum metric matching [28, 33, 39], Steiner tree [38], etc.).

A much more general benchmark is the *dynamic Min-Sum Set Cover* problem, where the algorithm is compared against an optimal solution allowed to change permutations over time. This problem has not been studied even in the offline case. In this work, we define the problem formally and obtain first results for the online case.

We remark that the online dynamic MSSC problem belongs to a rich class of problems called

Metrical Task Systems (MTS) [15]. MTS is a far-reaching generalization of several fundamental online problems and provides a unified framework for studying online problems (we discuss this in more detail in Section 1.2). Indeed, our results suggest that solving the online dynamic MSSC requires the development of powerful generic techniques for online problems, which might have further implications for the broader setting of MTS.

Throughout this paper, whenever we refer to online problems, like Min-Sum Set Cover or list update, we assume the static case, unless stated otherwise.

Previous Work on List Update. Prior to our work, the only version of online MSSC studied is the special case where $|S_t| = 1$ for all sets; this is the celebrated list update problem and it has been extensively studied (an excellent reference is [14]). It is known that the deterministic competitive ratio it least $2 - \frac{2}{n+1}$ and there are several 2-competitive algorithms known; most notably, the Move-to-Front (MTF) algorithm, which moves the (unique) element of S_t to the first position of the permutation, and the Frequency Count algorithm, which orders the elements in decreasing order according to their frequencies.

The dynamic list update problem has also been extensively studied. MTF is known to be 2-competitive [43] and there are several other 2-competitive algorithms [1,24].

1.1 Our Results

In this work, we initiate a systematic study of the online Min-Sum Set Cover problem. We consider the *r*-uniform case, where all request sets have the same size $|S_t| = r$. This is without loss of generality, as we explain in Section 1.3.

The first of our main results is a tight bound on the deterministic competitive ratio of Online MSSC. We show that the competitive ratio of deterministic algorithms is $\Omega(r)$.

Theorem 1.1. Any deterministic online algorithm for the Online Min-Sum Set Cover problem has competitive ratio at least $(r+1)(1-\frac{r}{n+1})$.

Note that for r = 1, this bound evaluates to $2 - \frac{2}{n+1}$, which is exactly the best known lower bound for the list update problem.

We complement this result by providing a matching (up to constant factors) upper bound.

Theorem 1.2. There exists a (5r+2)-competitive deterministic online algorithm for the Online Min-Sum Set Cover problem.

Interestingly, all prior work on the list update problem (case r = 1) does not seem to provide us with the right tools for obtaining an algorithm with such guarantees! As we discuss in Section 2, virtually all natural generalizations of successful list update algorithms (e.g., Moveto-Front, Frequency Count) end up with a competitive ratio way far from the desired bound. In fact, even for r = 2, most of them have a competitive ratio depending on n, such as $\Omega(\sqrt{n})$ or even $\Omega(n)$.

This suggests that online MSSC has a distinctive combinatorial structure, very different from that of list update, whose algorithmic understanding calls for significant new insights. The main reason has to do with the disjunctive nature of the definition of the access cost $\pi(S_t)$. In list update, where r = 1, the optimal solution is bound to serve a request S_t by its unique element. The only question is how fast an online algorithm should upgrade it (and the answer is "as fast as possible"). In MSSC, the hard (and crucial) part behind the design of any competitive algorithm is how to ensure that the algorithm learns fast enough about the element e_t used by the optimal solution to serve each request S_t . This is evident in the highly adaptive nature of the deceptively simple greedy algorithm of [25] and in the adversarial request sequences for generalizations of Move-to-Front, in Section 2. To obtain the asymptotically optimal ratio of Theorem 1.2, we develop a rounding scheme and use it to derandomize the multiplicative weights update (MWU) algorithm. Our analysis bounds the algorithm's access cost in terms of the optimal cost, but it does not account for the algorithm's moving cost. We then refine our approach, by performing lazy updates to the algorithm's permutation, and obtain a competitive algorithm for online MSSC.

We also observe (in Section 1.3) that based on previous work of Blum and Burch [12], there exists a (computationally inefficient) randomized algorithm with competitive ratio $1 + \epsilon$, for any $\epsilon \in (0, 1/4)$. This implies that no lower bound is possible, if randomization is allowed, and gives a strong separation between deterministic and randomized algorithms.

Memoryless Algorithms. While the bounds of Theorems 1.1 and 1.2 are matching, our algorithm from Theorem 1.2 is computationally inefficient since it simulates the MWU algorithm, which in turn, maintains a probability distribution over all n! permutations. This motivates the study of trade-offs between the competitive ratio and computational efficiency. To this end, we propose a memoryless algorithm, called *Move-All-Equally* (MAE), which moves all elements of set S_t towards the beginning of the permutation at the same speed until the first reaches the first position. This is inspired by the Double Coverage algorithm from k-server [20, 21]. We believe that MAE achieves the best guarantees among all memoryless algorithms. We show that this algorithm can not match the deterministic competitive ratio.

Theorem 1.3. The competitive ratio of the Move-All-Equally algorithm is $\Omega(r^2)$.

Based on Theorem 1.3, we conjecture that an O(r) guarantee cannot be achieved by a memoryless algorithms. We leave as an open question whether MAE has a competitive ratio f(r), or a dependence on n is necessary. To this end, we show that the competitive ratio of MAE is at most $2^{O(\sqrt{\log n \cdot \log r})}$ (see Section 4 for details).

Dynamic Min-Sum Set Cover. We also consider the dynamic version of online MSSC. Dynamic MSSC is much more general and the techniques developed for the static case do not seem adequately powerful. This is not surprising, since the MWU algorithm is designed to perform well against the best static solution. We investigate the performance of the MAE algorithm. First, we obtain an upper bound on its competitive ratio.

Theorem 1.4. The competitive ratio of the Move-All-Equally algorithm for the dynamic online Min-Sum Set Cover problem is $O(r^{3/2}\sqrt{n})$.

Although this guarantee is not very strong, we show that, rather surprisingly, it is essentially tight and no better guarantees can be shown for this algorithm.

Theorem 1.5. For any $r \ge 3$, the competitive ratio of the Move-All-Equally algorithm for the dynamic online Min-Sum Set Cover problem is $\Omega(r\sqrt{n})$.

This lower bound is based on a carefully crafted adversarial instance; this construction reveals the rich structure of this problem and suggests that more powerful generic techniques are required in order to achieve any f(r) guarantees. In fact, we conjecture that the lower bound of Theorem 1.1 is the best possible (ignoring constant factors) even for the dynamic problem and that using a work-function based approach such a bound can be obtained.

1.2 Further Related Work

Multiple Intents Re-ranking. This is a generalization of MSSC where for each set S_t , there is a *covering requirement* $K(S_t)$, and the cost of covering a set S_t is the position of the $K(S_t)$ -th element of S_t in π . The MSSC problem is the special case where $K(S_t) = 1$ for all sets S_t .

Another notable special case is the Min-Latency Set Cover problem, which corresponds to the other extreme case where $K(S_t) = |S_t|$ [29]. Multiple Intents Re-ranking was first studied by Azar et. al. [5], who presented a $O(\log r)$ -approximation; later O(1)-approximation algorithms were obtained [10,32,42]. Further generalizations have been considered, such as the Submodular Ranking problem, studied by Azar and Gamzu [4], which generalizes both Set Cover and MSSC, and the Min-Latency Submodular Cover, studied by Im et.al [31].

Prediction from Expert Advice and Randomized MSSC. In prediction from expert advice, there are N experts and expert i incurs a cost c_i^t in each step. A learning algorithm decides which expert i_t to follow (before the cost vector \mathbf{c}^t is revealed) and incurs a cost of $c_{i_t}^t$. The landmark technique for solving this problem is the multiplicative weights update (MWU - a.k.a. Hedge) algorithm. For an in-depth treatment of MWU, we refer to [3,27,35].

In the classic online learning setting, there is no cost for moving probability mass between experts. However, in a breakthrough result, Blum and Burch [12] showed that MWU is $(1 + \epsilon)$ competitive against the best expert, even if there is a cost D for moving probability mass between experts. By adapting this result to online MSSC (regarding permutations as experts), we can get an (inefficient) randomized algorithm with competitive ratio $(1 + \epsilon)$, for any constant $\epsilon \in (0, 1/4)$. A detailed description is deferred to the full version of this paper.

Metrical Task Systems and Online Dynamic MSSC. The online dynamic Min-Sum Set Cover problem belongs to a rich family of problems called Metrical Task Systems (MTS). In MTS, we are given a set of N states and a metric function d specifying the cost of moving between the states. At each step, a task arrives; the cost of serving the task at state i is c_i . An algorithm has to choose a state to process the task. If it switches from state i to state j and processes the task there, it incurs a cost $d(i, j) + c_j$. Given an initial state and a sequence of requests, the goal is to process all tasks at minimum cost.

It is easy to see that the online version of dynamic MSSC problem is a MTS, where the states correspond to permutations, thus N = n!, and the distance between two states is their Kendall tau distance. For a request set S_t , the request is a vector specifying the cost $\pi(S_t)$ for every permutation π .

Several other fundamental online problems (e.g., k-server, convex body chasing) are MTS. Although there has been a lot of work on understanding the structure of MTS problems [2, 8, 9, 15, 16, 22, 34, 40, 41], there is not a good grasp on how the structure relates to the hardness of MTS problems. Getting a better understanding on this area is a long-term goal, since it would lead to a systematic framework for solving online problems.

1.3 Preliminaries

Notation. Given a request sequence $S = \{S_1, \ldots, S_m\}$, for any algorithm ALG we denote Cost(ALG(S)) or simply Cost(ALG) the total cost of ALG on S. Similarly we denote AccessCost(ALG) the total access cost of ALG and MovingCost(ALG) the total movement cost of ALG. For a particular time step t, an algorithm using permutation π_t incurs an access $cost(ALG(t)) = \pi_t(S_t)$. We denote by $\pi_t[j]$ the position of element $j \in U$ in the permutation π_t .

Online Min-Sum Set Cover. We focus on the *r*-uniform case, i.e., when all sets S_t have size $r \ll n$. This is essentially without loss of generality, because we can always let $r = \max_t |S_t|$ and add the $r - |S_t|$ last unrequested elements in the algorithm's permutation to any set S_t with $|S_t| < r$. Assuming that $r \leq n/2$, this modification cannot increase the optimal cost and cannot decrease the online cost by more than a factor of 2.

2 Lower Bounds on the Deterministic Competitive Ratio

We start with a lower bound on the deterministic competitive ratio of online MSSC.

Theorem 1.1. Any deterministic online algorithm for the Online Min-Sum Set Cover problem has competitive ratio at least $(r+1)(1-\frac{r}{n+1})$.

For the proof, we employ an averaging argument, similar to those in lower bounds for list update and k-server [36, 43]. In each step, the adversary requests the last r elements in the algorithm's permutation. Hence, the algorithm's cost is at least (n - r + 1). Using a counting argument, we show that for any fixed set S_t of size r and any $i \in [n - r + 1]$, the number of permutations π with access $\cot \pi(S_t) = i$ is $\binom{n-i}{r-1}r!(n-r)!$. Summing up over all permutations and dividing by n!, we get that the average access $\cot f$ or S_t is $\binom{n+1}{r+1}\frac{r!(n-r)!}{n!} = \frac{n+1}{r+1}$. Therefore, the cost of the optimal permutation is a most $\frac{(n+1)}{r+1}$, and the competitive ratio of the algorithm at least $\frac{(n-r+1)(r+1)}{n+1}$. The details can be found in Appendix A.

Lower Bounds for Generalizations of Move-to-Front. For list update, where r = 1, simple algorithms like Move-to-Front (MTF) and Frequency Count achieve an optimal competitive ratio. We next briefly describe several such generalizations of them and show that their competitive ratio depends on n, even for r = 2. Missing details can be found in Appendix A. **MTF**_{first}: Move to the first position (of the algorithm's permutation) the element of S_t appearing first in π_t . This algorithm is $\Omega(n)$ -competitive when each request S_t consists of the last two elements in π_t . Then, the last element in the algorithm's permutation never changes and is used by the optimal permutation to serve the entire sequence!

MTF_{last}: Move to the first position the element of S_t appearing last in π_t .

MTF_{all}: Move to the first r positions all elements of S_t (in the same order as in π_t).

 $\mathbf{MTF}_{\mathbf{random}}$: Move to the first position an element of S_t selected uniformly at random.

MTF_{last}, MTF_{all} and MTF_{random} have a competitive ratio of $\Omega(n)$ when each request S_t consists of a fixed element e (always the same) and the last element in π_t , because they all incur an (expected for MTF_{random}) moving cost of $\Theta(n)$ per request.

The algorithms seen so far fail for the opposite reasons: MTF_{first} cares only about the first element and ignores completely the second, and the others are very aggressive on using the second (*r*th) element. A natural attempt to balance those two extremes is the following.

MTF_{relative}: Let *i* be the position of the first element of S_t in π_t . Move to the first positions of the algorithm's permutation (keeping their relative order) all elements of S_t appearing up to the position $c \cdot i$ in π_t , for some constant *c*. The bad instance for this algorithm is when each request S_t consists of the last element and the element at position $\lfloor n/c \rfloor - 1$ in π_t ; it never uses the *n*th element and the adversary serves all requests with it at a cost of 1.

All generalizations of MTF above are memoryless and they all fail to identify the element by which optimal serves S_t . The following algorithm tries to circumvent this by keeping memory and in particular the frequencies of reqested elements.

 $\mathbf{MTF}_{\mathbf{count}}$: Move to the first position the most frequent element of S_t (i.e., the element of S_t appearing in most requested sets so far).

This algorithm behaves better in easy instances, however with some more work we can show a lower bound of $\Omega(\sqrt{n})$ on its competitive ratio. Let e_1, \ldots, e_n be the elements indexed according to the initial permutation π_0 and $b = \sqrt{n}$. The request sequence proceeds in m/n phases of length n each. The first n - b requests of each phase are $\{e_1, e_2\}, \{e_1, e_3\}, \ldots, \{e_1, e_{n-b}\}$, and the last b requests consist of e_{n-b+i} and the element at position n-b at the current algorithm's permutation, for $i = 1, \ldots, b$. An optimal solution can cover all the requests by the elements $e_1, e_{n-b+1}, \ldots, e_n$ with total cost $\Theta(m+n\sqrt{n})$. The elements e_{n-b+1}, \ldots, e_n are never upgraded by MTF_{count}. Hence, the algorithm's cost is $\Theta(m\sqrt{n})$.

3 An Algorithm with Asymptotically Optimal Competitive Ratio

Next, we present algorithm Lazy-Rounding (Algorithm 2) and analyze its competitive ratio. The following is the main result of this section:

Theorem 1.2. Deterministic online algorithm Lazy-Rounding, presented in Algorithm 2, is (5r + 2)-competitive for the static version of the Online Min-Sum Set Cover problem.

The remainder of this section is devoted to the proof of Theorem 1.2. At a high-level, our approach is summarized by the following three steps:

- 1. We use as black-box the multiplicative weights update (MWU) algorithm with learning rate $1/n^3$. Using standard results from learning theory, we show that its expected access cost is within a factor 5/4 of OPT, i.e., AccessCost(MWU) $\leq \frac{5}{4}$ Cost(OPT) (Section 3.1).
- 2. We develop an online rounding scheme, which turns any randomized algorithm \mathcal{A} into a deterministic one, denoted Derand(\mathcal{A}), with access cost at most $2r \cdot \mathbb{E}[\operatorname{AccessCost}(\mathcal{A})]$ (Section 3.2). However, our rounding scheme does not provide any immediate guarantee on the moving cost of Derand(\mathcal{A}).
- 3. Lazy-Rounding is a lazy version of Derand(MWU) that updates its permutation only if MWU's distribution has changed a lot. A *phase* corresponds to a time interval that Lazy-Rounding does not change its permutation. We show that during a phase:
 - (i) The upper bound on the access cost increases, compared to Derand(MWU), by a factor of at most 2, i.e., $AccessCost(Lazy-Rounding) \leq 4r \cdot \mathbb{E}[AccessCost(MWU)]$ (Lemma 3.6).
 - (ii) The (expected) access cost of MWU is at least n^2 . Since our algorithm moves only once per phase, its movement cost is at most n^2 . Thus we get that (Lemma 3.7):

 $MovingCost(Lazy-Rounding) \leq \mathbb{E}[AccessCost(MWU)].$

For the upper bound on the moving cost above, we relate how much MWU's distribution changes during a phase, in terms of the total variation distance, to the cost of MWU and the cost of our algorithm.

Based on the above properties, we compare the access and the moving cost of Lazy-Rounding against the access cost of MWU and to get the desired competitive ratio:

 $Cost(Lazy-Rounding) \le (4r+1) \mathbb{E}[AccessCost(MWU)] \le (5r+2) Cost(OPT).$

Throughout this section we denote by $d_{\text{TV}}(\delta, \delta')$ the total variation distance of two discrete probability distributions $\delta, \delta' : [N] \to [0, 1]$, defined as $d_{\text{TV}}(\delta, \delta') = \sum_{i=1}^{N} \max\{0, \delta(i) - \delta'(i)\}$.

3.1 Using Multiplicative Weights Update in Online Min-Sum Set Cover

In this section, we explain how the well-known MWU algorithm [27,35] is used in our context. **The MWU Algorithm.** Given n! permutations of elements of U, the algorithm has a parameter $\beta \in [0, 1]$ and a weight w_{π} for each permutation $\pi \in [n!]$, initialized at 1. At each time step the algorithm chooses a permutation according to distribution $\mathbf{P}_{\pi}^{t} = w_{\pi}^{t}/(\sum_{\pi \in [n!]} w_{\pi}^{t})$. When request S_{t} arrives, MWU incurs an expected access cost of

$$\mathbb{E}[\operatorname{AccessCost}(\operatorname{MWU}(t))] = \sum_{\pi \in [n!]} \mathbf{P}_{\pi}^{t} \cdot \pi(S_{t})$$

and updates its weights $w_{\pi}^{t+1} = w_{\pi}^t \cdot \beta^{\pi(S_t)}$, where $\beta = e^{-1/n^3}$; this is the so-called *learning rate* of our algorithm. Later on, we discuss the reasons behind choosing this value.

On the Access Cost of MWU. Using standard results from learning theory [27, 35] and adapting them to our setting, we get that the (expected) access cost of MWU is bounded by Cost(OPT). This is formally stated in Lemma 3.1 (and is proven in Appendix B).

Lemma 3.1. For any request sequence $\sigma = (S_1, \ldots, S_m)$ we have that

$$\mathbb{E}[\operatorname{AccessCost}(\mathrm{MWU})] \leq \frac{5}{4} \cdot \operatorname{Cost}(\mathrm{OPT}) + 2n^4 \ln n \,.$$

On the Distribution of MWU. We now relate the expected access cost of the MWU algorithm to the total variation distance among MWU's distributions. More precisely, we show that if the total variation distance between MWU's distributions at times t_1 and t_2 is large, then MWU has incurred a sufficiently large access cost. The proof of the following makes a careful use of MWU's properties and is deferred to Appendix B.

Lemma 3.2. Let P^t be the probability distribution of the MWU algorithm at time t. Then,

$$d_{\mathrm{TV}}(\mathbf{P}^t, \mathbf{P}^{t+1}) \le \frac{1}{n^3} \cdot \mathbb{E}[\mathrm{AccessCost}(\mathrm{MWU}(t))].$$

The following is useful for the analysis of Lazy-Rounding. Its proof follows from Lemma 3.2 and the triangle inequality and is deferred to Appendix B.

Lemma 3.3. Let t_1 and t_2 two different time steps such that $d_{\text{TV}}(\mathbf{P}^{t_1}, \mathbf{P}^{t_2}) \geq 1/n$. Then,

$$\sum_{t=t_1}^{t_2-1} \mathbb{E}[\operatorname{AccessCost}(\operatorname{MWU}(t))] \ge n^2.$$

3.2 Rounding

Next, we present our rounding scheme. Given as input a probability distribution δ over permutations, it outputs a fixed permutation ρ such that for each possible request set S of size r, the cost of ρ on S is within a O(r) factor of the expected cost of the distribution δ on S. For convenience, we assume that n/r is an integer. Otherwise, we use $\lceil n/r \rceil$.

Algorithm 1 Greedy-Rounding (derandomizing probability distributions over the permutations)

Input: A probability distribution δ over [n!]. **Output:** A permutation $\rho \in [n!]$.

1: $\mathbf{R} \leftarrow U$ 2: for i = 1 to n/r do 3: $S^i \leftarrow \arg\min_{S \in \{R\}^r} \mathbb{E}_{\pi \sim \delta}[\pi(S)]$ 4: Place the elements of S^i (arbitrarily) from positions $(i-1) \cdot r + 1$ to $i \cdot r$ of ρ . 5: $R \leftarrow R \setminus S^i$ 6: end for 7: return ρ Our rounding algorithm is described in Algorithm 1. At each step, it finds the request S with minimum expected covering cost under the probability distribution δ and places the elements of S as close to the beginning of the permutation as possible. Then, it removes those elements from set R and iterates. The main claim is that the resulting permutation has the following property: any request S of size r has covering cost at most O(r) times of its expected covering cost under the probability distribution δ .

Theorem 3.4. Let δ be a distribution over permutations and let ρ be the permutation output by Algorithm 1 on δ . Then, for any set S, with |S| = r,

$$\rho(S) \le 2r \cdot \mathop{\mathbb{E}}_{\pi \sim \delta}[\pi(S)] \,.$$

Proof Sketch. The key step is to show that if the element used by ρ to serve the request S was picked during the kth iteration of the rounding algorithm, then $\mathbb{E}_{\pi \sim \delta}[\pi(S)] \geq k/2$. Clearly, $\rho(S) \leq k \cdot r$ and the theorem follows. Full proof is in Appendix B.2.

3.3 The Lazy Rounding Algorithm

Lazy-Rounding, presented in Algorithm 2, is essentially a lazy derandomization of MWU. At each step, it calculates the distribution on permutations maintained by MWU. At the beginning of each *phase*, it sets its permutation to that given by Algorithm 1. Then, it sticks to the same permutation for as long as the total variation distance of MWU's distribution at the beginning of the phase to the current MWU distribution is at most 1/n. As soon as the total variation distance exceeds 1/n, Lazy-Rounding starts a new phase.

The main intuition behind the design of our algorithm is the following. In Section 3.2 we showed that Algorithm 1 results in a deterministic algorithm with access cost no larger than $2r \mathbb{E}[\text{AccessCost}(\text{MWU})]$. However, such an algorithm may incur an unbounded moving cost; even small changes in the distribution of MWU could lead to very different permutations after rounding. To deal with that, we update the permutation of Lazy-Rounding only if there are substantial changes in the distribution of MWU. Intuitively, small changes in MWU's distribution should not affect much the access cost (this is formalized in Lemma 3.5). Moreover, Lazy-Rounding switches to a different permutation only if it is really required, which we use to bounds Lazy-Rounding's moving cost.

Algorithm 2 Lazy Rounding

Input: Sequence of requests (S_1, \ldots, S_m) and the initial permutation $\pi_0 \in [n!]$. **Output:** A permutation π_t at each round t, which serves request S_t .

1: start-phase $\leftarrow 0$ 2: $\mathbf{P}^1 \leftarrow$ uniform distribution over permutations 3: for each round $t \ge 1$ do if $d_{tv}(P^t, P^{\text{start-phase}}) \leq 1/n$ then 4: $\pi_t \leftarrow \pi_{t-1}$ 5:6: else $\pi_t \leftarrow \text{Greedy-Rounding}(\mathbf{P}^t)$ 7: start-phase $\leftarrow t$ 8: end if 9: Serve request S_t using permutation π_t . $w_{\pi}^{t+1} = w_{\pi}^t \cdot e^{-\pi(S_t)/n^3}$, for all permutations $\pi \in [n!]$. $\mathbf{P}^{t+1} \leftarrow \text{Distribution on permutations of MWU}, \mathbf{P}_{\pi}^{t+1} = w_{\pi}^t/(\sum_{\pi \in [n!]} w_{\pi}^t).$ 10:11: 12:13: end for

Bounding the Access Cost. We first show that the access cost of Lazy-Rounding is within a factor of 4r from the expected access cost of MWU (Lemma 3.6). To this end, we first show that if the total variation distance between two distributions is small, then sampling from those distributions yields roughly the same expected access cost for any request S. The proof of the following is based on the optimal coupling lemma and can be found in Appendix B.3.

Lemma 3.5. Let δ and δ' be two probability distributions over permutations. If that $d_{\text{TV}}(\delta, \delta') \leq 1/n$, for any request set S of size r, we have that

$$\mathbb{E}_{\pi \sim \delta'}[\pi(S)] \le 2 \cdot \mathbb{E}_{\pi \sim \delta}[\pi(S)].$$

We are now ready to upper bound the access cost of our algorithm.

Lemma 3.6. AccessCost(Lazy-Rounding) $\leq 4r \cdot \mathbb{E}[\text{AccessCost}(\text{MWU})].$

Proof. Consider a phase of Lazy-Rounding starting at time t_1 . We have that at any round $t \ge t_1$, $\pi_t = \text{Greedy-Rounding}(\mathbf{P}^{t_1})$, as long as $d_{\text{TV}}(\mathbf{P}^t, \mathbf{P}^{t_1}) \le 1/n$. By Theorem 3.4 and Lemma 3.5, we have that,

AccessCost(Lazy-Rounding(t)) = $\pi_t(S_t) \leq 2r \cdot \underset{\pi \sim \mathbf{P}^{t_1}}{\mathbb{E}} [\pi(S_t)] \leq 4r \underset{\pi \sim \mathbf{P}^t}{\mathbb{E}} [\pi(S_t)].$

Overall we get, AccessCost(Lazy-Rounding) = $\sum_{t=1}^{m} \pi_t(S_t) \leq 4r \mathbb{E}[AccessCost(MWU)].$

Bounding the Moving Cost. We now show that the moving cost of Lazy-Rounding is upper bounded by the expected access cost of MWU.

Lemma 3.7. MovingCost(Lazy-Rounding) $\leq \mathbb{E}[\operatorname{AccessCost}(MWU)].$

Proof. Lazy-Rounding moves at the end of a phase incurring a cost of at most n^2 . Let t_1 and t_2 be the starting times of two consecutive phases. By the definition of Lazy-Rounding, $d_{\text{TV}}(P^{t_1}, P^{t_2}) > 1/n$. By Lemma 3.3, we have that the access cost of MWU during t_1 and t_2 is at least n^2 . We get that

$$\frac{\text{MovingCost(ALG)}}{\mathbb{E}[\text{AccessCost(MWU)}]} \le \frac{n^2 \# \text{ different phases}}{n^2 \# \text{ different phases}} = 1. \quad \Box$$

Theorem 1.2 follows from lemmas 3.6, 3.7 and 3.1. The details can be found in Appendix B.3.

Remark. Note that to a large extent, our approach is generic and can be used to provide static optimality for a wide range of online problems. The only requirement is that there is a maximum access cost C_{max} and a maximum moving cost D; then, we should use MWU with learning rate $1/(D \cdot C_{max})$ and move when $d_{TV} \ge 1/C_{max}$. Here we used $D = n^2$ and $C_{max} = n$. The only problem-specific part is the rounding of Section 3.2. We explain the details in Appendix B. We believe it is an interesting direction to use this technique for generalizations of this problem, like multiple intents re-ranking or interpret known algorithms for other problems like the BST problem using our approach.

4 A Memoryless Algorithm

In this section we focus on memoryless algorithms. We present an algorithm, called Move-All-Equally (MAE), which seems to be the "right" memoryless algorithm for online MSSC. MAE decreases the index of all elements of the request S_t at the same speed until one of them reaches the first position of the permutation (see Algorithm 3). Note that MAE belongs to the *Move-to-Front* family, i.e., it is a generalization of the classic MTF algorithm for the list update problem. MAE admits two key properties that substantially differentiate it from the other algorithms in the Move-to-Front family presented in Section 2.

Algorithm 3 Move-All-Equally

Input: A request sequence (S_1, \ldots, S_m) and the initial permutation $\pi_0 \in [n!]$ **Output:** A permutation π_t at each round t.

- 1: for each round $t \ge 1$ do
- 2: $k_t \leftarrow \min\{i \mid \pi_{t-1}[i] \in S_t\}$
- 3: Decrease the index of all elements of S_t by $k_t 1$.
- 4: **end for**
- (i) Let e_t denote the element used by OPT to cover the request S_t . MAE always moves the element e_t towards the beginning of the permutation.
- (ii) It balances moving and access costs: if the access cost at time t is k_t , then the moving cost of MAE is roughly $r \cdot k_t$ (see Algorithm 3). The basic idea is that the moving cost of MAE can be compensated by the decrease in the position of element e_t . This is why it is crucial all the elements to be moved with the same speed.

Lower Bound. First, we show that this algorithm, besides its nice properties, fails to achieve a tight bound for the online MSSC problem.

Theorem 1.3. The competitive ratio of the Move-All-Equally algorithm is $\Omega(r^2)$.

In the lower bound instance, the adversary always requests the last r elements of the algorithm's permutation. Since MAE moves all elements to the beginning of the permutation, we end up in a request sequence where n/r disjoint sets are repeatedly requested. Thus the optimal solution incurs a cost of $\Theta(n/r)$ per request, while MAE incurs a cost of $\Omega(n \cdot r)$ per request (the details are in Appendix C). Note that in such a sequence, MAE loses a factor of rby moving all elements, instead of one. However, this extra movement seems to be the reason that MAE outperforms all other memoryless algorithms and avoids poor performance in trivial instances, like other MTF-like algorithms.

Upper Bounds. Let \mathcal{L} denote the set of elements used by the optimal permutation on a request sequence such that $|\mathcal{L}| = \ell$. That means, OPT has those ℓ elements in the beginning of its permutation, and it never uses the remaining $n - \ell$ elements. Consider a potential function $\Phi(t)$ being the number of inversions between elements of \mathcal{L} and $U \setminus \mathcal{L}$ in the permutation of MAE (an inversion occurs when an element of \mathcal{L} is behind an element of $U \setminus \mathcal{L}$). Consider the request S_t at time t and let k_t be the access cost of MAE.

Let e_t be the element used by OPT to serve S_t . Clearly, in the permutation of MAE, e_t passes (i.e., changes relative order w.r.t) $k_t - 1$ elements. Among them, let L be the set of elements of \mathcal{L} and R the elements of $U \setminus \mathcal{L}$. Clearly, $|L| + |R| = k_t - 1$ and $|L| \leq |\mathcal{L}| = \ell$. We get that the move of e_t changes the potential by -|R|. The moves of all other elements increase the potential by at most $(r-1) \cdot \ell$. We get that

$$k_t + \Phi(t) - \Phi(t-1) \leq |L| + |R| - |R| + (r-1) \cdot \ell \leq |L| + (r-1) \cdot \ell \leq r \cdot \ell$$

Since the cost of MAE at step t is no more than $(r + 1) \cdot k_t$, we get that the amortized cost of MAE per request is $O(r^2 \cdot \ell)$. This implies that for all sequences such that OPT uses all elements of \mathcal{L} with same frequencies (i.e, the OPT pays on average $\Omega(\ell)$ per request), MAE incurs a cost within $O(r^2)$ factor from the optimal. Recall that all other MTF-like algorithms are $\Omega(\sqrt{n})$ competitive even in instances where OPT uses only one element!

While this simple potential gives evidence that MAE is $O(r^2)$ -competitive, it is not enough to provide satisfactory competitiveness guarantees. We generalize this approach and define the potential function $\Phi(t) = \sum_{j=1}^{n} \alpha_j \cdot \pi_t(j)$, where $\pi_t(j)$ is the position of element j at round tand α_j are some non-negative coefficients. The potential we described before is the special case where $\alpha_j = 1$ for all elements of \mathcal{L} and $\alpha_j = 0$ for elements of $U \setminus \mathcal{L}$.

By refining further this approach and choosing coefficients α_j according to the frequency that OPT uses element j to serve requests (elements of high frequency are "more important" so they should have higher values α_j), we get an improved upper bound.

Theorem 4.2. The competitive ratio of MAE algorithm is at most $2^{O(\sqrt{\log n \cdot \log r})}$.

Note that this guarantee is $o(n^{\epsilon})$ and $\omega(\log n)$. The proof is based on the ideas sketched above but the analysis is quite involved and is deferred to Appendix C.

5 Dynamic Online Min-Sum Set Cover

In this section, we turn our attention to the dynamic version of online MSSC. In online dynamic MSSC, the optimal solution maintains a trajectory of permutations $\pi_0^*, \pi_1^*, \ldots, \pi_t^*, \ldots$ and use permutation π_t^* to serve each request S_t . The cost of the optimal dynamic solution is $OPT_{dynamic} = \sum_t (\pi_t^*(S_t) + d_{KT}(\pi_{t-1}^*, \pi_t^*))$, where $\{\pi_t^*\}_t$ denotes the optimal permutation trajectory for the request sequence that minimizes the total access and moving cost.

We remark that the ratio between the optimal static solution and the optimal dynamic solution can be as high as $\Omega(n)$. For example, in the sequence of requests $\{1\}^b \{2\}^b \dots \{n\}^b$, the optimal static solution pays $\Theta(n^2b)$, whereas the optimal dynamic solution pays $\Theta(n^2 + n \cdot b)$ by moving the element that covers the next $n \cdot b$ requests to the first position and then incurring access cost 1. The above example also reveals that although Algorithm 2 is $\Theta(r)$ -competitive against the optimal static solution, its worst-case ratio against a dynamic solution can be $\Omega(n)$.

MAE Algorithm. As a first study of the dynamic problem, we investigate the competitive ratio of *Move-All-Equally* (MAE) algorithm from Section 4. We begin with an upper bound.

Theorem 1.4. The competitive ratio of the Move-All-Equally algorithm for the dynamic online Min-Sum Set Cover problem is $O(r^{3/2}\sqrt{n})$.

The approach for proving Theorem 1.4 is generalizing that exhibited in Section 4 for the static case. We use a generalized potential function $\Phi(t) = \sum_{j=1}^{n} \alpha_j^t \cdot \pi_t(j)$; i.e, the multipliers α_j may change over time so as to capture the moves of $OPT_{dynamic}$. To select coefficients α_j^t we apply a two-level approach. We observe that there is always a 2-approximate optimal solution that moves an element of S_t to the front (similar to classic MTF in list update). We call this MTF_{OPT} . We compare the permutation of the online algorithm with the permutation maintained by this algorithm; at each time, elements the beginning of the offline permutation are considered to be "important" and have higher coefficients α_j^t . The formal proof is in Appendix D.

Next, we show an almost matching lower bound.

Theorem 1.5. For any $r \ge 3$, the competitive ratio of the Move-All-Equally algorithm for the dynamic online Min-Sum Set Cover problem is $\Omega(r\sqrt{n})$.

Sketch of the Construction. The lower bound is based on a complicated adversarial request sequence; we sketch the main ideas. Let k be an integer. During a *phase* we ensure that:

(i) There are 2k "important" elements used by OPT; we call them e_1, \ldots, e_{2k} . In the beginning of the phase, those elements are ordered in the start of the optimal permutation π^* , i.e., $\pi^*[e_j] = j$. The phase contains k consecutive requests to each of them, in order; thus the total number of requests is $\approx 2k^2$. OPT brings each element e_j at the front and uses

it for k consecutive requests; thus the access cost of OPT is $2k^2$ (1 per request) and the total movement cost of OPT of order $\Theta(k^2)$. Over a phase of $2k^2$ requests, OPT incurs an overall cost $\Theta(k^2)$, i.e., an average of O(1) per request.

(ii) The first k + r - 2 positions of the online permutation will be always occupied by the same set of "not important" elements; at each step the r - 2 last of them will be part of the request set and MAE will move them to the front. Thus the access cost will always be k + 1 and the total cost more than $(r + 1) \cdot k$.

The two properties above are enough to provide a lower bound $\Omega(r \cdot k)$; the optimal cost is O(1) per request and the online cost $\Omega(r \cdot k)$. The goal of an adversary is to construct a request sequence with those two properties for the largest value of k possible.

The surprising part is that although MAE moves all requested elements towards the beginning of the permutation, it never manages to bring any of the "important" elements in a position smaller than r+k-2. While the full instance is complex and described in Appendix D, at a high-level, we make sure that whenever a subsequence of k consecutive requests including element e_j begins, e_j is at the end of the online permutation, i.e., $\pi_t[e_j] = n$. Thus, even after k consecutive requests where MAE moves it forward by distance k, it moves by k^2 positions; by making sure that $n - k^2 > r + k - 2$ (which is true for some $k = \Omega(\sqrt{n})$), we can make sure that e_j does not reach the first r + k - 2 positions of the online permutation.

6 Concluding Remarks

Our work leaves several intriguing open questions. For the (static version of) Online MSSC, it would be interesting to determine the precise competitive ratio of the MAE algorithm; particularly whether it depends only on r or some dependency on n is really necessary. More generally, it would be interesting to determine the best possible performance of memoryless algorithms and investigate trade-offs between competitiveness and computational efficiency.

For the online dynamic MSSC problem, the obvious question is whether a f(r)-competitive algorithm is possible. Here, we showed that techniques developed for the list update problem seem to be too problem-specific and are not helpful in this direction. This calls for the use of more powerful and systematic approaches. For example, the online primal-dual method [18] has been applied successfully for solving various fundamental problems [6,7,17,19]. Unfortunately, we are not aware of a primal-dual algorithm even for the special case of list update; the only attempt we are aware of is in [45], but this analysis basically recovers known (problem-specific) algorithms using dual-fitting. Our work gives further motivation for designing a primal-dual algorithm for list-update: this could be a starting point towards solving the online dynamic MSSC.

In a broader context, the online MSSC is the first among a family of poorly understood online problems such as the multiple intents re-ranking problem described in Section 1.2. In this problem, when a set S_t is requested, we need to cover it using $s \leq r$ elements; MSSC is the special case s = 1. It is natural to expect that the lower bound of Theorem 1.1 can be generalized to $\Omega(r/s)$, i.e., as s grows, we should be able to achieve a better competitive ratio. It will be interesting to investigate this and the applicability of our technique to obtain tight bounds for this problem.

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A Deferred Proofs of Section 2

In this section, we include the proofs deferred from section 2. First, we prove the general lower bound for the competitive ratio of any deterministic online algorithm for the Online Min Sum Set Cover problem.

Theorem 1.1. Any deterministic online algorithm for the Online Min-Sum Set Cover problem has competitive ratio at least $(r + 1) \cdot (1 - \frac{r}{n+1})$.

Proof. Let ALG be any online algorithm. The adversary creates a request sequence in which every request is composed by the r last elements of the current permutation of ALG. At each round t, ALG incurs an accessing cost of at least n - (r - 1). Thus for the whole request sequence of m requests, $Cost(ALG) \ge m \cdot (n - r + 1)$.

The non-trivial part of the proof is to estimate the cost of the optimal static permutation. We will count total cost of all n! static permutations and use the average cost as an upper bound on the optimal cost. For any request set S_t , we intend to find the total cost of the n!permutations for S_t . To do this, we will count the number permutations that have access cost of i, for every $1 \le i \le n - (r - 1)$. For such counting, there are two things to consider. First, in how many different ways we can choose the positions where the r elements of S_t are located and second how many different orderings on elements of S_t and of $U \setminus S_t$ exist. We address those two separately.

- (i) For a permutation π that incurs an access cost of *i*, it follows that, from the elements in S_t , the first one in π is located in position *i* and no other element from the set is located in positions j < i. The other r 1 elements of S_t are located among the last n i positions of π . There are $\binom{n-i}{r-1}$ different ways to choose the locations of those elements.
- (ii) Once the positions of elements of S_t have been fixed, there are r! different ways to assign the elements in those positions, equal to the number of permutations on r elements. Similarly, there are (n-r)! different ways to assign elements of $U \setminus S_t$ to the n-r remaining positions.

Gathering the above, we conclude that the number of permutations that incur access cost exactly i for a fixed request S_t is

$$\binom{n-i}{r-1}r!(n-r)!$$

The latter implies two basic facts:

- 1. $\sum_{i=1}^{n-r+1} {n-i \choose r-1} r! (n-r)! = n!$ (since each permutation has a specific cost for request S_t).
- 2. The total sum of access costs for fixed request of size r is:

Total-Access-Cost =
$$\sum_{i=1}^{n-r+1} i \cdot {\binom{n-i}{r-1}} r! (n-r)!$$

=
$$\sum_{i=1}^{n-r+1} \sum_{j=i}^{n-r+1} {\binom{n-j}{r-1}} r! (n-r)!$$
 By reordering the terms
=
$$\sum_{i=1}^{n-r+1} \sum_{j=i}^{n-r+1} {\binom{n-j}{r-1}} r! (n-r)!$$

permutations with access cost $\ge i$
=
$$\sum_{i=1}^{n-r+1} {\binom{n-i+1}{r}} r! (n-r)!$$

$$= r!(n-r)!\binom{n+1}{r+1}$$

where the last equality follows by the fact that $\sum_{i=1}^{n-r+1} {n-i \choose r-1} r! (n-r)! = n!$ (see number 1 above) with $n \leftarrow n+1$ and $r \leftarrow r+1$. Hence for a request sequence of length m, we get that

$$\operatorname{Cost}(\operatorname{OPT}) \le m \cdot \frac{r!(n-r)!}{n!} \binom{n+1}{r+1} = m \cdot \frac{n+1}{r+1}$$

We conclude that for any deterministic algorithm ALG, we have:

$$\frac{\operatorname{Cost}(\operatorname{ALG})}{\operatorname{Cost}(\operatorname{OPT})} \ge \frac{m \cdot (n-r+1)}{m \cdot \frac{n+1}{r+1}} = (r+1) \cdot \left(1 - \frac{r}{n+1}\right) = r+1 - \frac{r(r+1)}{n+1}. \quad \Box$$

Lower bounds for various algorithms. Next, we prove the lower bounds for the competitive ratio of the several online algorithms generalizing the MTF algorithm. For all lower bounds we use request sets of size r = 2. Recall that $\pi_t(j)$ the *j*th element of the permutation π_t for $1 \le j \le n$

 $\mathbf{MTF}_{\mathbf{first}}$: Move to the first position (of the algorithm's permutation) the element of S_t appearing first in π_t .

Lower bound: Let the request sequence $S_1, S_2, \ldots S_m$, in which S_t contains the last two elements of MTF_{first}'s permutation at round t-1. Formally, $S_t = \{\pi_t(n-1), \pi_t(n)\}$. MTF_{first} moves the first element of the request in the first position and in every round the last element in MTF_{first}'s permutation remains the same $(\pi_t(n) = \pi_0(n))$. As a result, MTF_{first} pays $\Omega(n)$ in each request, whereas OPT has the element $\pi_0[n]$ in the first position and just pays 1 per request.

MTF_{last}: Move to the first position the element of S_t appearing last in π_t .

Lower bound: Let the request sequence $S_1, S_2, \ldots S_m$, in which each set S_t always contains the last element of $\pi_t - 1$ and the fixed element 1. Clearly MTF_{last} pays $\Omega(m \cdot n)$, while cost(OPT) = m by having element 1 in the first place.

MTF_{all}: Move to the first r positions all elements of S_t (in the same order as in π_t). Lower bound: The same as previous.

 $\mathbf{MTF}_{\mathbf{random}}$: Move to the first position an element of S_t selected uniformly at random.

Lower bound: Let the request sequence S_1, S_2, \ldots, S_m , in which each set S_t always contains an element selected uniformly at random from π_t and the fixed element 1. Therefore elements in the last n/2 positions of π_t have probability 1/2 to be chosen. At each round t, MTF_{random} moves with probability 1/2 to the first position of the list, the element of S_t that was randomly selected. Thus at each round t, MTF_{random} pays with probability 1/4, moving cost at least n/2, meaning that the overall expected cost is at least $m \cdot n/8$. As a result, the ratio is $\Omega(n)$ since OPT pays m by keeping element 1 in the first position.

MTF_{relative}: Let *i* be the position of the first element of S_t in π_t . Move to the first positions of the algorithm's permutation (keeping their relative order) all elements of S_t appearing up to the position $c \cdot i$ in π_t , for some constant *c*.

Lower bound: Let the request sequence S_1, \ldots, S_n in which S_t contains the $\lfloor \frac{n-1}{c} \rfloor$ th and the *n*th element of the list at round t-1. MTF_{relative} never moves the last element and thus $\pi_n(0)$ belongs in all sets S_t . As in first case, this provides an $\Omega(n)$ ratio.

 $\mathbf{MTF}_{\mathbf{count}}$: Move to the first position the most frequent element of S_t (i.e., the element of S_t appearing in most requested sets so far).

Lower bound: The request sequence S_1, \ldots, S_m is specifically constructed so that MTF_{count} never moves the last b elements of the initial permutation π_0 .

$$\pi_0 = [\underbrace{x_1, \dots, x_{n-b}}_{n-b \text{ elements}}, \underbrace{x_{n-b+1}, \dots, x_n}_{b \text{ elements}}]$$

The constructed request sequence S_1, \ldots, S_m will be composed by m/n sequences of length n. Each piece of length n will have the following form:

- 1. n b requests $\{x_1, x_2\}, \{x_1, x_3\}, \dots, \{x_1, x_{n-b}\}$ (all requests contain x_1).
- 2. {element in position $n b, x_{n-b+i}$ } for i = 1 to b (additional b requests).

After the requests of type 1, the list is the same as the initial one, since x_1 has frequency n - band $x_2, \ldots x_{n-b}$ have frequency 1. Now consider the requests of type 2. MTF_{count} moves always to the front the element which is in position n - b, since has already been involved in a type 1 request and has greater frequency. Therefore, MTF_{count} pays $b \cdot (n - b)$. Repeating the same request sequence m/n times, we can construct a sequence of length m. In this request sequence, OPT keeps the element x_1 in the first position and the elements $\{x_{n-b+1}, \ldots, x_n\}$ in the next b positions. Thus, OPT pays $(n - b) \cdot m/n$ for the requests of type 1 and $b^2 \cdot m/n$ for the requests of type 2. MTF_{count} pays $(n - b) \cdot m/n$ for the requests of type 1 (same as OPT), but $(n - b) \cdot b \cdot m/n$ for the requests of type 2. Setting $b = \sqrt{n}$, we get a $\Omega(\sqrt{n})$ lower bound for the competitive ratio of MTF_{count}, which concludes this section.

B Deferred Proofs of Section 3

In this section we include the proofs deferred from Section 3.

B.1 Proofs Related to MWU Algorithm

Here we include omitted proofs related to MWU algorithm.

Access Cost of MWU. We first show that the MWU is 5/4-competitive for access costs. Lemma 3.1. For any request sequence $\sigma = (S_1, \ldots, S_m)$ we have that

$$\mathbb{E}[\operatorname{AccessCost}(\operatorname{MWU})] \leq \frac{5}{4} \cdot \operatorname{Cost}(\operatorname{OPT}) + 2n^4 \ln n.$$

Proof. By the standard results in learning theory [27, 35], we know that for any sequence $\sigma = (S_1, \ldots, S_m)$, the MWU algorithm satisfies

$$\sum_{t=1}^{m} \sum_{\pi \in [n!]} \mathbf{P}_{\pi}^{t} \cdot \operatorname{AccessCost}(\pi, S_{t}) \leq \frac{\ln(1/\beta)}{1-\beta} \cdot \min_{\pi \in [n!]} \sum_{t=1}^{m} \pi(S_{t}) + \frac{\ln(n!)}{1-\beta}$$

where $\beta = e^{-1/n^3}$. Thus, $3/4 < \beta < 1$, for any $n \ge 2$. Using standard inequalities we get that $\frac{\ln(1/\beta)}{1-\beta} \le 5/4$ and $1-\beta \ge 1/2n^3$ for any $n \ge 2$. We finally get that,

$$\mathbb{E}[\operatorname{AccessCost}(\mathrm{MWU})] \leq \frac{5}{4} \cdot \operatorname{Cost}(\mathrm{OPT}) + 2n^4 \ln n.$$

Total Variation Distance. We now proceed on the proofs of lemmas relating the total variation distance of the distribution maintained by the MWU algorithm to its access cost.

Lemma 3.2. Let P^t be the probability distribution of MWU algorithm at time t. Then, the probability distribution P^{t+1} of the algorithm satisfies

$$d_{\mathrm{TV}}(\mathbf{P}^t, \mathbf{P}^{t+1}) \le \frac{1}{n^3} \cdot \mathbb{E}[\mathrm{AccessCost}(\mathrm{MWU}(t))].$$

Proof. To simplify notation, let $W^t = \sum_{\pi \in [n!]} w_{\pi}^t$. We remind that by the definition of MWU, $w_{\pi}^{t+1} = w_{\pi}^t \cdot e^{-\pi(S_t)/n^3}$. Moreover, by the definition of total variation distance,

$$\begin{split} d_{\mathrm{TV}}(\mathbf{P}^{t},\mathbf{P}^{t+1}) &= \sum_{\pi:\mathbf{P}_{\pi}^{t} > \mathbf{P}_{\pi}^{t+1}} \mathbf{P}_{\pi}^{t} - \mathbf{P}_{\pi}^{t+1} = \sum_{\pi:\mathbf{P}_{\pi}^{t} > \mathbf{P}_{\pi}^{t+1}} \left(\frac{w_{\pi}^{t}}{W^{t}} - \frac{w_{\pi}^{t+1}}{W^{t+1}}\right) \\ &\leq \sum_{\pi:\mathbf{P}_{\pi}^{t} > \mathbf{P}_{\pi}^{t+1}} \left(\frac{w_{\pi}^{t}}{W^{t}} - \frac{w_{\pi}^{t+1}}{W^{t}}\right) \\ &\leq \sum_{\pi\in[n!]} \left(\frac{w_{\pi}^{t}}{W^{t}} - \frac{w_{\pi}^{t+1}}{W^{t}}\right) = \sum_{\pi\in[n!]} \frac{w_{\pi}^{t}}{W^{t}} \cdot \left(1 - e^{-\pi(S_{t})/n^{3}}\right) \\ &= \sum_{\pi\in[n!]} \mathbf{P}_{\pi}^{t} \cdot \left(1 - e^{-(\pi(S_{t})/n^{3})}\right) \leq \sum_{\pi\in[n!]} \mathbf{P}_{\pi}^{t} \cdot \frac{\pi(S_{t})}{n^{3}} \\ &= \frac{1}{n^{3}} \cdot \mathbb{E}[\operatorname{AccessCost}(\mathrm{MWU}(t))] \,. \end{split}$$

In the first inequality we used that $W^{t+1} \leq W^t$. In the second inequality we used that for all π we have that $w_{\pi}^{t+1} \leq w_{\pi}^t$ which implies that $\frac{w_{\pi}^t - w_{\pi}^{t+1}}{W^t} \geq 0$. In the last inequality we used that $1 - e^x \leq -x$, for any x.

Lemma 3.3. Let t_1 and t_2 two different time steps such that $d_{\text{TV}}(\mathbf{P}^{t_1}, \mathbf{P}^{t_2}) \geq 1/n$. Then, during the time interval $[t_1, t_2)$ the cost of the MWU algorithm is at least n^2 .

$$\sum_{t=t_1}^{t_2-1} \mathbb{E}[\operatorname{AccessCost}(\operatorname{MWU}(t))] \ge n^2.$$

Proof. By Lemma 3.2 and summing over all t such that $t_1 \leq t < t_2$, we have that

$$\sum_{t=t_1}^{t_2-1} d_{\mathrm{TV}}(\mathbf{P}^t, \mathbf{P}^{t+1}) \le \frac{1}{n^3} \cdot \sum_{t=t_1}^{t_2-1} \mathbb{E}[\operatorname{AccessCost}(\mathrm{MWU}(t))].$$
(1)

By triangle inequality we have that $d_{\text{TV}}(\mathbf{P}^{t_1}, \mathbf{P}^{t_2}) \leq \sum_{t=t_1}^{t_2-1} d_{\text{TV}}(\mathbf{P}^t, \mathbf{P}^{t+1})$. Combined with (1), this implies that

$$d_{\mathrm{TV}}(\mathbf{P}^{t_1}, \mathbf{P}^{t_2}) \le \frac{1}{n^3} \sum_{t=t_1}^{t_2-1} \mathbb{E}[\mathrm{AccessCost}(\mathrm{MWU}(t))].$$

By rearranging and using that $d_{\text{TV}}(\mathbf{P}^{t_1}, \mathbf{P}^{t_2}) \geq 1/n$, we get that

$$\sum_{t=t_1}^{t_2-1} \mathbb{E}[\operatorname{AccessCost}(\operatorname{MWU}(t))] \ge n^3 \cdot \frac{1}{n} = n^2 \qquad \qquad \square$$

B.2 Rounding

Theorem 3.4.Let δ be a distribution over permutations and let ρ be the permutation output by Algorithm 1 on δ . Then, for any set S, with |S| = r,

$$\rho(S) \le 2r \cdot \mathop{\mathbb{E}}_{\pi \sim \delta}[\pi(S)] \,.$$

Proof. Let e be the element used by ρ to serve the request on set S. Pick k such that $(k-1) \cdot r + 1 \leq \operatorname{AccessCost}(\rho, S) \leq k \cdot r$. That means, e was placed at the permutation ρ at the kth iteration of the rounding algorithm.

Let $S^1, \ldots, S^{n/r}$ be the sets chosen during the rounding algorithm. Recall that ρ uses an element from S^k to serve the request. To this end, we use the technical Lemma B.1 in order to get a lower bound on the expected cost of δ . We distinguish between two cases:

- 1. Case $S = S^k$. In that case, by Lemma B.1 we get that $\mathbb{E}_{\pi \sim \delta}[\pi(S)] \geq \frac{k+1}{2}$.
- 2. Case $S \neq S^k$. That means, e is one element of S in S_k and no elements of S are in sets S_1, \ldots, S_{k-1} . By construction of or rounding algorithm, we have that

$$\mathop{\mathbb{E}}_{\pi \sim \delta}[\pi(S)] \ge \mathop{\mathbb{E}}_{\pi \sim \delta}[\pi(S_k)] \ge \frac{k+1}{2}.$$

We get that in both cases $\mathbb{E}_{\pi \sim \delta}[\pi(S)] \geq \frac{k+1}{2}$. We conclude that

$$\frac{\operatorname{AccessCost}(\rho, S)}{\mathbb{E}_{\pi \sim \delta}[\operatorname{AccessCost}(\delta, S)]} \le \frac{k \cdot r}{\frac{k+1}{2}} \le 2 \cdot r \quad \Box$$

We now proceed to the lemma omitted in the proof of Theorem 3.4.

Lemma B.1. Let δ be a probability distribution over permutations and $1 \leq k \leq \frac{n}{r}$. Let S_1, \ldots, S_k be disjoint sets such that $S_j \subseteq U$ and $|S_j| = r$ for all $1 \leq j \leq k$. Let $\mathcal{E}_j = \mathbb{E}_{\pi \sim \delta}[\pi(S_j)]$ for any $1 \leq j \leq k$. If $\mathcal{E}_1 \leq \ldots \leq \mathcal{E}_k$, then, we have that $\mathcal{E}_j \geq \frac{j+1}{2}$, for $1 \leq j \leq k$. *Proof.* We have that

where $\sum_{\ell=1}^{j} \pi(S_{\ell}) \geq \frac{j(j+1)}{2}$ follows by the fact that $\pi(S_{\ell})$ take *j* different positive integer values (the sets S_{ℓ} are disjoint).

B.3 Our Algorithm

In this section we include all proofs related to the analysis of our algorithm. **Lemma 3.5** Let δ and δ' be two probability distributions over permutations. If that $d_{\text{TV}}(\delta, \delta') \leq 1/n$, for any request set S of size r, we have that

$$\mathbb{E}_{\pi \sim \delta'}[\pi(S)] \le 2 \cdot \mathbb{E}_{\pi \sim \delta}[\pi(S)]$$

Proof. By coupling lemma there exists a coupling (X, Y) such that

$$\Pr[X \neq Y] = d_{\rm TV}(\delta, \delta'). \tag{2}$$

Clearly, $\mathbb{E}[\operatorname{AccessCost}(X, S)] = \mathbb{E}_{\pi \sim \delta}[\pi(S)]$ and $\mathbb{E}[\operatorname{AccessCost}(Y, S)] = \mathbb{E}_{\pi \sim \delta'}[\pi(S)]$. Note that since minimum cost for serving a request is 1 and maximum n, it will always hold that

$$1 \leq \mathbb{E}[\operatorname{AccessCost}(X, S)], \mathbb{E}[\operatorname{AccessCost}(Y, S)] \leq n.$$

We will show that $\mathbb{E}[\operatorname{AccessCost}(Y, S) - \operatorname{AccessCost}(X, S)] \leq 1$. This implies the lemma as follows:

$$\begin{split} \mathbb{E}_{\pi \sim \delta'}[\pi(S)] &= \mathbb{E}[\operatorname{AccessCost}(Y, S)] \\ &\leq \mathbb{E}[\operatorname{AccessCost}(X, S)] + 1 \leq 2 \mathbb{E}[\operatorname{AccessCost}(X, S)] \\ &= 2 \mathbb{E}_{\pi \sim \delta}[\pi(S)]. \end{split}$$

Thus it remains to show that $\mathbb{E}[\operatorname{AccessCost}(Y, S) - \operatorname{AccessCost}(X, S)] \leq 1$. For notational convenience, let the random variable $Z = \operatorname{AccessCost}(X, S) - \operatorname{AccessCost}(Y, S)$. It suffices to show that $\mathbb{E}[Z] \leq 1$. We have that

$$\mathbb{E}[Z] \le \Pr[\operatorname{AccessCost}(X, S) = \operatorname{AccessCost}(Y, S)] \cdot 0 + \Pr[\operatorname{AccessCost}(X, S) \neq \operatorname{AccessCost}(Y, S)] \cdot n,$$
(3)

since whenever $X \neq Y$, the difference in the cost is upper bounded by n.

Observe that $\Pr[X(S) \neq Y(S)] \leq \Pr[X \neq Y]$; this is because if X = Y, then $\operatorname{AccessCost}(X, S) = \operatorname{AccessCost}(Y, S)$, but it may happen that $X \neq Y$ but $\operatorname{AccessCost}(X, S) = \operatorname{AccessCost}(Y, S)$. From (3) we get that

$$\mathbb{E}[Z] \le \Pr[X \neq Y] \cdot n. \tag{4}$$

Combining (2) with (4) we get that

$$\mathbb{E}[Z] \le d_{\mathrm{TV}}(\delta, \delta') \cdot n \le \frac{1}{n} \cdot n = 1.$$

Proof of Theorem 1.2. We now give the formal proof of the competitive ratio of Lazy-Rounding algorithm.

Theorem 1.2. There exists a (5r+2)-competitive deterministic online algorithm for the static version of the Online Min-Sum Set Cover problem.

Proof. We show that our algorithm ALG is (5r + 2)-competitive.

By lemmata 3.6 and 3.7 we get that

$$Cost(Lazy-Rounding) \le (4r+1) \cdot AccessCost(MWU).$$
(5)

Now, we connect the access cost of MWU to the optimal cost. By Lemma 3.1 we have that

AccessCost(MWU)
$$\leq \frac{5}{4} \cdot \text{Cost}(\text{OPT}) + 2n^4 \ln n.$$
 (6)

By (5) and (6) we get that

$$Cost(Lazy-Rounding) \le (5r+2) \cdot Cost(OPT) + 2 \cdot (4r+1) \cdot n^4 \ln n \qquad \Box$$

On our Approach. We now show how our approach for static optimality can be used to a wide class of online problems.

Consider any MTS problem with N states such that in each request the service cost of each state is at most C_{max} and the diameter of the state space is D. Assume that there exists a rounding scheme providing a derandomization of MWU such that the service cost is within α factor of the expected service cost of MWU. We explain how our technique from Section 3 can be used to obtain a $O(\alpha)$ -competitive algorithm against the best state.

Algorithm. The algorithm is essentially the same as Algorithm 2, using the MWU algorithm with learning rate $1/(D \cdot C_{\text{max}})$ and moving when the total variation distance between the distributions exceeds $1/C_{\text{max}}$.

This way, Lemma 3.2 would give a bound of

$$d_{\mathrm{TV}}(\mathbf{P}^t, \mathbf{P}^{t+1}) \le \frac{1}{D} \cdot \operatorname{AccessCost}(\mathrm{MWU}(t))$$

and as a consequence in Lemma 3.3 we will get that the (expected) cost of MWU during a phase is at least D. Since the algorithm moves only at the end of the phase, incurring a cost of at most D, Lemma 3.7 will still hold. Last, it is easy to see that Lemma 3.5 will then hold for $d_{\rm TV}(\delta, \delta') \leq 1/C_{\rm max}$. This way, Lemma 3.6 would give that

 $\operatorname{AccessCost}(\operatorname{ALG}) \leq 2\alpha \mathbb{E}[\operatorname{Cost}(\operatorname{MWU})].$

Combining the above, we get that the algorithm is $O(\alpha)$ -competitive.

C Deferred Proofs of Section 4

In this section, we provide the proofs omitted from section 4. More specifically, we prove a lower bound and an an upper bound on the competitive ratio of the MAE algorithm against the static optimal solution.

C.1 Lower Bound

We start with the lower bound of $\Omega(r^2)$ on the competitive ratio of MAE against OPT.

Theorem 1.3. The competitive ratio of the Move-All-Equally algorithm is $\Omega(r^2)$.

Proof. MAE is given an initial permutation π_0 with size n, where $n = r \cdot k$, for integers r, k both greater than 1. At each round t, the adversary gives requests S_t , which consist of the last r elements in the permutation π_{t-1} of MAE. Since MAE moves all the elements of S_t to the beginning of π_t , the request S_{t+1} contains the r elements preceding the elements of S_t in π_t . Thus, the request sequence can be divided in m/k requests containing the same k pairwise disjoint requests, denoted as $S^* = \{S_1^*, \ldots, S_k^*\}$. The optimal static solution can serve the request sequence by using only k elements, where each of these elements belongs to exactly one of the

 $S_j^*, 1 \leq j \leq k$. OPT has these elements in the first k positions and pays $1 + 2 + \ldots + k \leq k^2$ for every k consecutive requests (S_1^*, \ldots, S_k^*) . MAE clearly pays n - r + 1 access cost and $r \cdot (n - r + 1)$ moving cost on every request, therefore MAE = $m \cdot (r + 1) \cdot (n - r + 1)$. Then, the competitive ratio of MAE is at least

$$\frac{\operatorname{Cost}(\operatorname{MAE})}{\operatorname{Cost}(\operatorname{OPT})} \ge \frac{m \cdot (r+1) \cdot (n-r+1)}{(m/k) \cdot k^2} = \frac{(r+1) \cdot (r \cdot k - r + 1)}{k} = \Omega(r^2) \quad \Box$$

C.2 Upper Bound

We now proceed to the proof of the upper bound on the competitive ratio of MAE algorithm.

Theorem 4.2. The competitive ratio of MAE algorithm is at most $2^{O(\sqrt{\log n \cdot \log r})}$.

Let the frequency of an element e be the fraction of requests served by e in the optimal permutation. At a high-level, we divide the optimal permutation into k+1 blocks where in each of the first k blocks the frequencies of all elements are within a factor of β and the last block contains all elements with frequencies at most $1/n^2$. By construction, in worst-case $k \approx \frac{\log n}{\log \beta}$.

The main technical contribution of this section is proving the following lemma.

Lemma C.3. For any parameter $\beta > 1$,

$$\operatorname{Cost}(\operatorname{MAE}) \le 4\beta(2r)^{k+1} \cdot \operatorname{Cost}(\operatorname{OPT}) + (2r)^k n^2$$

where $k = \lceil 2 \log n / \log \beta \rceil$.

Note that Theorem 4.2 follows from Lemma C.3 by balancing the values of β and $(2r)^{k+1}$. It is easy to verify that by setting $\beta = 2^{\sqrt{8 \log n \cdot \log 2r}}$, we obtain a competitive ratio at most $\beta^2 < 2^{8\sqrt{\log n \log r}}$

Notation and Definitions. Let $f_j \in [0, 1]$ denote the *covering frequency*, which is the total fraction of requests served by the optimal permutation with element j. For convenience, we reorder the elements such that $f_1 \ge f_2 \ge \ldots \ge f_n$. As a result, $\text{Cost}(\text{OPT}) = m \cdot \sum_{j=1}^n j \cdot f_j$.

We partition the elements of U into k + 1 blocks, as follows. The last block, E_{k+1} contains all elements j with $f_j \leq 1/n^2$; intuitively those are the least important elements. The block E_i for $1 \leq i \leq k$ contains all elements with frequencies in the interval $[f_1/\beta^{i-1}, f_1/\beta^i)$. Note that in worst case, $k = \lceil 2 \log n / \log \beta \rceil$; we need that $f_1/\beta^k \leq 1/n^2$, which is equivalent to $k \geq f_1 \cdot \frac{2 \log n}{\log \beta}$.

Lower Bound on optimal cost. Using this structure, we can express a neat lower bound on the cost of the optimal static permutation, which is formally stated in the following lemma. Lemma C.4 provides a lower bound on Cost(OPT) using the first k blocks. For the rest of this section, let $\mathcal{F}_i = \bigcup_{j=1}^i E_j$ the set of all elements in the first i blocks, for $1 \leq i \leq k$. Let also f_{\max}^i, f_{\min}^i denote the maximum and minimum covering frequencies in the set E_i .

Lemma C.4. For any request sequence the optimal cost is at least

$$\operatorname{Cost}(\operatorname{OPT}) \ge \frac{m}{2} \sum_{i=1}^{k} |E_i| \cdot |\mathcal{F}_i| \cdot f_{\min}^i.$$

Proof. Using the definitions above, OPT uses element j exactly $m \cdot f_j$ times for a total cost $j \cdot m \cdot f_j$. To account for all elements of the same bucket E_i together, we underestimate this

cost and for an element $j \in E_i$ we charge OPT only for $m \cdot f_{\min}^i \leq m \cdot f_j$ requests. We get that:

$$Cost(OPT) = m \sum_{j=1}^{n} f_{j} \cdot j \ge m \sum_{i=1}^{k} \sum_{j \in E_{i}} f_{j} \cdot j \ge m \sum_{i=1}^{k} f_{\min}^{i} \sum_{j \in E_{i}} j$$
$$= m \sum_{i=1}^{k} f_{\min}^{i} \sum_{j=1}^{|E_{i}|} (j + |\mathcal{F}_{i-1}|) = m \sum_{i=1}^{k} f_{\min}^{i} \left(|E_{i}||\mathcal{F}_{i-1}| + \frac{|E_{i}|(|E_{i}| + 1)}{2} \right)$$
$$\ge \frac{m}{2} \sum_{i=1}^{k} f_{\min}^{i} \cdot |E_{i}| \cdot (|\mathcal{F}_{i-1}| + |E_{i}|) = \frac{m}{2} \sum_{i=1}^{k} f_{\min}^{i} \cdot |E_{i}| \cdot |\mathcal{F}_{i}| \quad \Box$$

Upper Bound on cost of MAE. Using the blocks, we also obtain an upper bound on the cost of Move-All-Equally. This is formally stated in Lemma C.5.

Lemma C.5. For any request sequence the cost of MAE algorithm can be upper bounded as follows:

Cost(MAE)
$$\leq (2r)^{k+1} \cdot m \sum_{i=1}^{k} |\mathcal{F}_i| |E_i| \cdot f_{\max}^i + 2(2r)^{k+1} \cdot m + (2r)^k \cdot n^2.$$

Before proceeding to the proof of Lemma C.5, which is quite technically involved, we show how Lemma C.3 follows from Lemmata C.5 and C.4.

Proof of Lemma C.3. From Lemma C.5 we have that

$$\operatorname{Cost}(\operatorname{MAE}) \leq (2r)^{k+1} \cdot \underbrace{m \sum_{i=1}^{k} |\mathcal{F}_i| |E_i| \cdot f_{\max}^i}_{\leq \beta \cdot \operatorname{Cost}(\operatorname{OPT})} + \underbrace{2(2r)^{k+1} \cdot m}_{\leq 2(2r)^{k+1} \cdot \operatorname{Cost}(\operatorname{OPT})} + (2r)^k \cdot n^2$$
$$\leq 4\beta (2r)^{k+1} \cdot \operatorname{Cost}(\operatorname{OPT}) + (2r)^k \cdot n^2,$$

where in the inequality we used the lower bound on Cost(OPT) from Lemma C.4 and that $f_{\text{max}}^i = f_{\text{min}}^i \cdot \beta$ for all *i*.

Proof of Lemma C.5

It remains to prove Lemma C.5 which gives the upper bound on the cost of MAE algorithm. The proof lies on the right selections of the coefficients α_i in the potential function $\Phi(t)$:

$$\Phi(t) = \sum_{j=1}^{n} \alpha_j \cdot \pi_t(j),$$

where $\pi_t(j)$ is the position of element j at round t and α_j are some non-negative coefficients. More precisely, if $j \in E_i$ then $\alpha_j = (2r)^{k-i}$ for $i = 1, \ldots, k$ and $\alpha_j = 0$ if $j \in E_{k+1}$.

At time t, let k_t denote the access cost of MAE and let e_t be the element used by OPT to serve request S_t . Using the coefficients mentioned above, we can break the analysis into two different types of requests: (i) requests served by OPT using an element $e_t \in E_i$ for $1 \le i \le k$ and (ii) requests served by OPT using $e_t \in E_{k+1}$. At a high-level the first case is the one where the choice of coefficients is crucial; for the second, we show that the frequencies are so "small", such that even if MAE incurs an access cost of n, this does not affect the bound of Theorem 4.2.

We start with the first type of requests. We show the following lemma.

Lemma C.6. At time t, if $e_t \in E_i$, for $1 \le i \le k$, then we have that

AccessCost(MAE(t)) + $\Phi(t) - \Phi(t-1) \le (2r)^k \cdot |\mathcal{F}_i|$

For the second type of requests we have the following lemma.

Lemma C.7. The total amortized cost of MAE algorithm for all requests such that $e_t \in E_{k+1}$ is at most $((2r)^k + 1) \cdot m$.

The proofs of those Lemmas are at the end of this section. We now continue the proof of Lemma C.5.

Proof of Lemma C.5. We upper bound the total access cost of MAE, that is $\sum_{t=1}^{m} k_t$. Clearly, by construction of MAE, the total cost is at most $(r+1) \cdot \sum_{t=1}^{m} k_t$. We have that

$$\begin{split} \sum_{t=1}^{m} k_t + \Phi(t) - \Phi(t-1) &= \sum_{i=1}^{k} \left(\sum_{t:e_t \in E_i} k_t + \Phi(t) - \Phi(t-1) + \sum_{t:e_t \in E_{k+1}} k_t + \Phi(t) - \Phi(t-1) \right) \\ &\leq \sum_{i=1}^{k} \sum_{t:e_t \in E_i} (2r)^k \cdot |\mathcal{F}_i| + \left((2r)^k + 1 \right) \cdot m \\ &= \sum_{i=1}^{k} (2r)^k \cdot |\mathcal{F}_i| \cdot m \cdot \sum_{j \in E_i} f_j^t + \left((2r)^k + 1 \right) \cdot m \\ &\leq \sum_{i=1}^{k} (2r)^k \cdot |\mathcal{F}_i| \cdot m \cdot \sum_{j \in E_i} f_{\max}^i + \left((2r)^k + 1 \right) \cdot m \\ &= \sum_{i=1}^{k} (2r)^k \cdot |\mathcal{F}_i| |E_i| \cdot f_{\max}^i \cdot m + \left((2r)^k + 1 \right) \cdot m, \end{split}$$

where the first inequality comes from Lemmata C.6, C.7. Using $\Phi(0) \leq (2r)^{k-1}n^2$ we get

$$Cost(MAE) \le 2r \cdot \sum_{t=1}^{m} k_t \le 2r \left(\sum_{i=1}^{k} (2r)^k \cdot |\mathcal{F}_i| |E_i| \cdot f_{\max}^i \cdot m + \left((2r)^k + 1 \right) \cdot m + (2r)^{k-1} n^2 \right)$$
$$\le (2r)^{k+1} \sum_{i=1}^{k} \cdot |\mathcal{F}_i| |E_i| \cdot f_{\max}^i \cdot m + 2(2r)^{k+1} \cdot m + (2r)^k \cdot n^2 \qquad \Box$$

We conclude the Section with the proofs of lemmata C.6, C.7, which were omitted earlier.

Proof of Lemma C.6. At time t, the access cost of MAE is k_t . We have that

$$\begin{aligned} k_t + \Phi(t) - \Phi(t-1) &= k_t - \alpha_{e_t} \cdot k_t + \sum_{j \neq e_t} \alpha_j (\pi_t(j) - \pi_{t-1}(j)) \\ &\leq k_t \cdot (1 - \alpha_{e_t}) + (2r)^{k-1} \cdot |\mathcal{F}_i| \cdot r + \sum_{j \in U \setminus \mathcal{F}_i} \alpha_j (\pi_t(j) - \pi_{t-1}(j)), \end{aligned}$$

where the inequality follows from the fact that each element in \mathcal{F}_i can increase its position by at most r and that the maximum coefficient α_j is at most $(2r)^{k-1}$. We complete the proof by showing that

$$k_t \cdot (1 - \alpha_{e_t}) + \sum_{j \in U \setminus \mathcal{F}_i} \alpha_j(\pi_t(j) - \pi_{t-1}(j)) \le 0$$

If $e_t \in E_k$ then $\alpha_{e_t} = 1$ and $\alpha_j = 0$ for all $j \in U \setminus \mathcal{F}_k$, thus the left hand side equals 0 and the inequality holds. It remains to analyze the case where $e_t \in E_i$ and i < k.

$$k_t \cdot (1 - \alpha_{e_t}) + \sum_{j \in U \setminus \mathcal{F}_i} \alpha_j (\pi_t(j) - \pi_{t-1}(j)) \le k_t \cdot (1 - \alpha_{e_t}) + \sum_{j \in U \setminus \mathcal{F}_i} \frac{\alpha_{e_t}}{2r} \cdot |\pi_t(j) - \pi_{t-1}(j)|$$
$$\le k_t \cdot (1 - \alpha_{e_t}) + \frac{\alpha_{e_t}}{2r} \cdot r \cdot k_t$$
$$= k_t \cdot (1 - \alpha_{e_t} + \frac{\alpha_{e_t}}{2}) \le 0.$$

The last inequality is due to the fact that $\alpha_{e_t} \ge r$, thus $1 - \alpha_{e_t}/2$ is negative if $r \ge 2$.

Proof of Lemma C.7. We sum the total amortized cost over all time steps t such that $e_t \in E_{k+1}$. We have that

$$\sum_{t:e_t \in E_{k+1}} k_t + \Phi(t) - \Phi(t-1) \le \sum_{t:e_t \in E_{k+1}} n + (2r)^{k-1} \cdot n \cdot r \le \left((2r)^k + 1 \right) \cdot n \sum_{t:e_t \in E_{k+1}} 1 \le \left((2r)^k + 1 \right) \cdot n \cdot m \cdot \sum_{j \in E_{k+1}} f_j \le \left((2r)^k + 1 \right) \cdot n \cdot m \cdot \frac{|E_{k+1}|}{n^2} \le \left((2r)^k + 1 \right) \cdot m \qquad \square$$

D Deferred Proofs of Section 5

In this section we present the full proofs of Theorems 1.4 and 1.5.

Upper bound. We start with the upper bound on the competitive ratio of the MAE algorithm. Recall that we use the generalized potential function $\Phi(t) = \sum_{j=1}^{n} \alpha_j^t \cdot \pi_t(j)$ for the dynamic case, where the multipliers α_j change over time so as to capture the moves of $OPT_{dynamic}$.

Theorem 1.4. The competitive ratio of the Move-All-Equally algorithm for the dynamic online Min-Sum Set Cover problem is $O(r^{3/2}\sqrt{n})$.

Given the solution of $\text{OPT}_{\text{dynamic}}$, we construct the solution $Move-To-Front_{\text{OPT}}$ (MTF_{OPT}), that at each round t moves e_t (the element $\text{OPT}_{\text{dynamic}}$ uses to cover S_t) in the first position of the permutation. We compare the cost of MAE, with the cost of MTF_{OPT} , which is just a 2-approximation of $\text{OPT}_{\text{dynamic}}$ (for the same reason than MTF is 2-competitive for list update – see Lemma D.3 for a proof). More precisely, $\alpha_j^t = 1$ if the element j is in one of the first c positions of MTF_{OPT} 's permutation at round t and $\alpha_j^t = 0$ otherwise (the exact value of c will be determined later).

Recall that the access cost of MAE at round t is k_t . We have that at each round $t \ge 1$,

$$k_{t} + \Phi(t) - \Phi(t-1) = k_{t} + \sum_{j=1}^{n} \alpha_{j}^{t} \cdot \pi_{t}(j) - \sum_{j=1}^{n} \alpha_{j}^{t-1} \cdot \pi_{t-1}(j)$$

$$= (1 - \alpha_{e_{t}}^{t}) \cdot k_{t} + \sum_{j \neq e_{t}} \alpha_{j}^{t} \cdot (\pi_{t}(j) - \pi_{t-1}(j))$$

$$+ \sum_{j=1}^{n} (\alpha_{j}^{t} - \alpha_{j}^{t-1}) \cdot \pi_{t-1}(j)$$

$$\leq r \cdot c + \underbrace{\left(\sum_{j=1}^{n} \alpha_{j}^{t} - \alpha_{j}^{t-1}\right)}_{\text{bounded by MTFOPT}(t)} \cdot n, \qquad (7)$$

where in the last inequality we used that $\alpha_{e_t}^t = 1$ and that for each element, its position change from round t - 1 to round $t \pi_t(j) - \pi_{t-1}(j)$ can be at most r.

Lemma D.2. The term $\sum_{j=1}^{n} (\alpha_j^t - \alpha_j^{t-1})$, which appears in the difference $\Phi(t) - \Phi(t-1)$ of the potential function $\Phi(t)$ can be bounded by the moving cost of the MTF_{OPT}(t) as follows: $\sum_{j=1}^{n} (\alpha_j^t - \alpha_j^{t-1}) \leq \alpha_{e_t}^t - \alpha_{e_t}^{t-1} \leq \text{MovingCost}(\text{MTF}_{\text{OPT}}(t))/c.$

Proof. For the first part of the inequality, observe that from round t - 1 to round t, MTF_{OPT} only moves element e_t towards the beginning of the permutation. The latter implies, that for all elements $j \neq e_t$, $\alpha_j^t \leq \alpha_j^{t-1}$. For the second part of the inequality, observe that if $\alpha_{e_t}^{t-1} = 0$, then by the definition of the coefficients, e_t does not belong in the first c positions (of MTF_{OPT}'s permutation) at round t - 1. Since at round t, MTF_{OPT} moves e_t in the first position of the list, MovingCost(MTF_{OPT}(t)) $\geq c$

Notice that the overall access cost of both MAE and MTF_{OPT} is just m since at each round t both of the algorithms admit an element, covering S_t , in the first position of the permutation. As a result by setting $c = \sqrt{n/r}$ and applying Lemma D.2,

$$Cost(MAE) = AccessCost(MAE) + MovingCost(MAE)$$

= $m + r \sum_{t=1}^{m} k_t$
 $\leq m + 2r^{3/2}\sqrt{n} \sum_{t=1}^{m} MovingCost(MTF_{OPT}(t)) + \Phi(0) - \Phi(m)$
 $\leq 2r^{3/2}\sqrt{n} \cdot Cost(MTF_{OPT}) + \Phi(0) - \Phi(m) \leq 2r^{3/2}\sqrt{n} Cost(MTF_{OPT}).$

The first inequality follows from inequality (7). The last inequality follows from the fact that $\Phi(0) \leq \Phi(m)$ since at round 0 the permutations of MAE and MTF_{OPT} are the same. We complete the section with the proof of Lemma D.3, stating that Cost(MTF_{OPT}) $\leq 2 \cdot Cost(OPT_{dynamic})$.

Lemma D.3. For any sequence of requests (S_1, \ldots, S_m) ,

$$Cost(MTF_{OPT}) \le 2 \cdot Cost(OPT_{dynamic})$$

Proof. To simplify notation, let x_t and y_t respectively denote the permutations of MTF_{OPT} and OPT_{dynamic} at round t. We will use as potential the function $\Phi(t) = d_{KT}(x_t, y_t)$ i.e. the number of inverted pairs between the permutation x_t and y_t . Let L_t denote the set of elements that are on the left of e_t in permutation x_t and on the left of e_t in permutation y_t . Respectively, R_t denotes the set of elements that are on the left of e_t in permutation x_t , but on the right of e_t in permutation y_t . Clearly, at round t, MTF_{OPT} pays $L_t + R_t$ for moving cost and 1 for accessing cost. At same time, OPT_{dynamic} pays at least $L_t + 1$ for accessing cost and $d_{KT}(y_t, y_{t-1})$ for moving cost.

$$Cost(MTF_{OPT}) + \Phi(t) - \Phi(t-1) = L_t + R_t + 1 + d_{KT}(x_t, y_t) - d_{KT}(x_{t-1}, y_t) + d_{KT}(x_{t-1}, y_t) - d_{KT}(x_{t-1}, y_{t-1}) \leq L_t + R_t + 1 + (L_t - R_t) + d_{KT}(x_{t-1}, y_t) - d_{KT}(x_{t-1}, y_{t-1}) \leq 2 \cdot (L_t + 1) + d_{KT}(y_t, y_{t-1})$$

The first inequality follows by the definition of the of the sets R_t and L_t , while the second by the triangle inequality in the Kendall-tau distance $d_{KT}(\cdot, \cdot)$. The proof is completed by summing all over m and using the fact that $d_{KT}(x_0, y_0) = 0$.

Lower bound. Next, we prove the following theorem showing a lower bound on MAE, which nearly matches its upper bound.

Theorem 1.5. For any $r \ge 3$, the competitive ratio of the Move-All-Equally algorithm for the dynamic online Min-Sum Set Cover problem is $\Omega(r\sqrt{n})$.

The constructed request sequence (S_1, \ldots, S_m) will have the following properties:

- 1. The size of every request S_t is $r \geq 3$.
- 2. Every request forces the algorithm to pay an access cost of $\Omega(\sqrt{n})$.
- 3. The request sequence contains $\Omega(\sqrt{n})$ consecutive requests that share a common element, which we call the *pivot*.

The first two properties ensure that MAE will pay $\Omega(r\sqrt{n})$ moving cost on every request. The third property will enable the optimal dynamic solution $OPT_{dynamic}$ to pay 1 access cost for the consecutive requests that share the common element.

Before proceeding to formal details of the proof of Theorem 1.5, we will present the main ideas of the construction. To this end, let π_0 be the MAE's initial permutation and think of it as being divided into k + 2 blocks. The first block contains k + r - 1 elements and all other blocks contain k elements, therefore $n = k^2 + 2k + r - 1$. The "important" elements, which the optimal dynamic solution uses to *cover* all requests, are the elements of the second and last block initially. Meaning that the optimal dynamic solution can cover all requests using only the 2k "important" elements. Figure 1 depicts the structure of the initial permutation for k = 3and r = 3, where "important" elements are colored red and green.

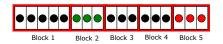


Figure 1: The structure in blocks for k = 3 and r = 3. Blocks are drawn with thick red borders. The elements that uses the optimal dynamic solution to cover the request sequence are colored green and red and lie in the second and last block initially.

The following definition formalizes the concept of blocks discussed above. Blocks are defined in terms of the indices of the permutation.

Definition D.5. Let $\pi = (\pi_0, \ldots, \pi_m)$ be a sequence of permutations, where each permutation has size $n = k^2 + 2k + r - 1$. We divide each permutation into k + 2 consecutive blocks $b_1, b_2, \ldots b_{k+2}$, where a block is a set of consecutive indices in each permutation with $|b_1| = k + r - 1$ and $|b_2| = |b_3| = \ldots = |b_{k+2}| = k$. Therefore, $b_1 = [1, \ldots, k + r - 1]$ and $b_i = [(i-1)k + r, \ldots, ik + r - 1]$, for $2 \le i \le k + 2$.

For ease of exposition, we will divide the analysis in phases, which are subdivided in rounds. The first round starts with the request S_1 , which contains the last element of the permutation π_0 (*pivot*₁) of the MAE algorithm and ends when *pivot*₁ arrives in position k + r. Inductively,

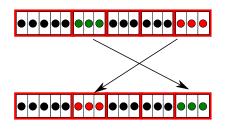


Figure 2: The permutation of MAE before the start of a round and after k rounds, for k = 3 and r = 3. The elements of the second block have moved to the last block and vice versa.

at the start of a round j, S_j contains the last element of permutation π_t and lasts until this element arrives in position k + r. A phase ends after 2k such rounds and a new phase begins. Formally:

Definition D.6. A phase defines the period, starting with k elements in block b_2 and k elements in block b_{k+2} and ending after 2k rounds of requests with these elements back in the blocks, where they started. A round j starts when the adversary requests the last element, denoted as pivot_j, of the current permutation of MAE (with index $k^2 + 2k + r - 1$) and ends when MAE places this element in the first position of the second block b_2 (with index k + r). The duration of the round is the number of requests from the start of the round until the end of the round and will be k + 1.

In order to describe the request sequence, we give the color green to k elements and the color red to k other. These are the elements, used by $OPT_{dynamic}$ and every request contains one of them. The rest of them are colored black. Let all the green elements be in the second block (b_2) and all red elements be in the last block (b_{k+2}) initially. The request sequence may seem complicated, however its constructed following two basic principles. The first is to decrease the position of the *pivot* element (which is initially red) by k in the permutation and the second is to increase the position of the k green elements by one on every request. This is easily achieved by requests S_t of the form:

$$S_t = \{\underbrace{\pi_t(k+1), \pi_t(k+2) \dots, \pi_t(k+r-2)}_{black}, \underbrace{\text{element succeeding the green block}}_{black}, \underbrace{\text{pivot}}_{red}\}$$

Then, at the end of round j, $pivot_j$ will be in the first position of the second block and the green elements will be in block j + 2. After k rounds, all red elements will be in the second block and all green elements will be in the last block, thus the adversary can repeat the request sequence starting again from round 1. Figure 2 depicts the positions of the "important" elements at the start of round 1 and at the end of round k.

The formal definition of the request sequence is shown in Algorithm 4.

Notice that the last request of round j places $pivot_j$ in the first position of block b_2 , just in front of pivot elements of previous rounds. This way it is guaranteed that all k red elements will be in b_2 after k rounds. Moreover, at each round j > 1 there is a request (the (k - j + 1)th) that increases the positions of k - j + 2 green elements by two, since they are passed from both the element succeeding them and the pivot element (the other j - 2 elements are passed only by the element succeeding the green block). Since, the adversary wants all green elements to be in block j + 2 after round j, the next request does not move the k - j + 2 green elements and the other j - 2 are moved because $pivot_j$ passes them (see Figure 3).

Algorithm 4 Adversarial request sequence

Let $S_{ij} = [p_1, \ldots, p_r]$ be the *i*-th request of round *j*, where p_1, \ldots, p_r denote the indices of the requested elements in the current permutation of MAE. The request sequence for *k* consecutive rounds is (every round has k + 1 requests): **Round 1.** The *i*th request of round 1, for $1 \le i < k$ is: $S_{i1} = [k + 1, \ldots, k + r - 2, 2k + r - 1 + i, k^2 + 2k + r - 1 - (i - 1)k]$ and the two last requests of round 1 are: $S_{k1} = [k + 1, \ldots, k + r - 1, 3k + r - 1]$ and $S_{(k+1)1} = [k, \ldots, k + r - 2, 2k + r - 1]$. **Round j from 2 to k.** The *i*th request of round *j*, for $i \notin \{k - j + 2, k + 1\}$ is : $S_{ij} = [k + 1, \ldots, k + r - 2, (j + 1)k + r - 1 + i, k^2 + 2k + r - 1 - (i - 1)k]$, for the request *i* with i = k - j + 2 is : $S_{ij} = [k + 1, \ldots, k + r - 1, (j + 1)k + r - 1]$ and for i = k + 1 is : $S_{(k+1)j} = [k, \ldots, k + r - 2, 2k + r - 1]$. Then, the request sequence starts again from round 1.

The following lemma shows that MAE will arrive in a symmetric permutation after k rounds.

Lemma D.7. Let π be a permutation of definition D.5 at the start of a round and let $X = \{x_1, x_2, \ldots, x_k\}$ and $Y = \{y_1, y_2, \ldots, y_k\}$ be the elements in b_2 and b_{k+2} respectively before the start of a round. Then, after k rounds, MAE has moved the elements of Y in block b_2 and the elements of X in block b_{k+2} .

Proof. We have to prove that after j rounds, j elements of Y will be in b_2 and all elements of X will be in b_{j+2} . Observe that the only way to increase the index of an element e in a permutation by c is to move c elements with higher index than e in the permutation to positions with lower index than e (it increases by one if an element arrives at the position of e).

Let $pivot_j$ be the pivot element of round j, which by definition is the last element in the permutation at the start of a round. After k requests to $pivot_j$ involving also the (k + 1)th element and elements in positions that do not increase the index of $pivot_j$, $pivot_j$ moves a total of k^2 positions to the left arriving in position 2k + r - 1 of the current permutation. Then, the last request of this round will move it to the first position of b_2 , moving the pivot element of the previous phase to the second position of b_2 . By construction of the request sequence, $pivot_j$ is the *j*th element of Y requested so far. Therefore, at the end of round j, j elements of Y will be in the first j positions of b_2 .

We now show that all elements of X will be in b_{j+2} at end of round j. Particularly, we show inductively that at the beginning of jth round all elements of X are in block b_{j+1} and MAE moves all elements of X to block b_{j+2} when the jth round ends.

Induction Base: In the first round, k requests contain the element succeeding X, r-2 elements in positions $k + 1, \ldots, k + r - 2$ of the permutation and an element, which has index higher than k + r - 2 (which is $pivot_1$) and does not change the positions of elements of X. The (k+1)th request does not change the positions of elements of X, since it contains elements with lower index. Therefore, the algorithm moves all elements of X exactly k positions to the right and they will be in b_3 when round 1 ends.

Inductive Step: For j > 1, we have k - 1 requests, where each of them forces the algorithm to move elements of X one position to the right. However, there is exactly one request S_{ij} with i = k - j + 1, where k - j + 2 elements of X increase their positions by two. These elements are passed by both the element succeeding X and the pivot element. The other j - 2 elements are passed only by the element succeeding X, thus increasing their positions by one. Therefore, the next request of the adversary (the k - j + 2-th) makes the algorithm move $pivot_j$ and the elements in positions k + 1, ..., k + r - 1, therefore moving only the j - 2 elements of X one position to the right and the other elements of X remain in their positions.

We conclude that at each round the elements of X move to the next right block. Since, they are initially positioned in b_2 , after k rounds they end up in block b_{k+2} .

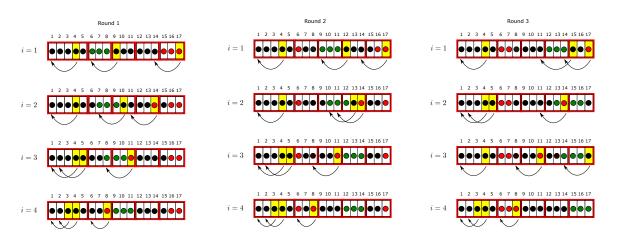


Figure 3: Execution of MAE on the adversarial sequence for r = 3 and k = 3. The requested positions in the list have yellow background and the arrows indicate the positions of the elements after every request. For round j > 1, the *i*th request with i = k - j + 1 makes MAE increase the position of k - j + 2 green elements by two. The next request is such that only the j - 2remaining green elements increase their position by one.

We are now ready to provide the proof of Theorem 1.5

Theorem 1.5. For any $r \ge 3$, the competitive ratio of the Move-All-Equally algorithm for the dynamic online Min-Sum Set Cover problem is $\Omega(r\sqrt{n})$.

Proof. We compute the costs paid by MAE and $OPT_{dynamic}$ after k rounds of a phase, since the request sequence of Definition 4 is then repeated.

First, we bound the optimal cost. Initially, the optimal solution incurs a moving cost $O(k \cdot n)$ to move the 2k elements of b_2 and b_{k+2} in the first 2k positions of its permutation. Then, at the start of round j, it brings $pivot_j$ to the first position and incurs an access cost of 1 for k + 1 consecutive requests. So, after k rounds it pays at most $2k^2$ moving cost plus $k \cdot (k+1)$ access cost, which sums to at most $4k^2$.

We now account the online cost. MAE pays $r \cdot k$ for the first k requests of each round and $r \cdot (k-1)$ for the last request. So, the total cost for k rounds is $r \cdot k \cdot (k^2 + k - 1) > r \cdot k^3$. From Lemma D.7, all elements of b_2 are in b_{k+2} and all elements of b_{k+2} are in b_2 after k rounds.

The adversary can repeat the same strategy to create an arbitrarily long request sequence. Let l be the number of times the same k-round strategy is applied. We get that

$$\frac{\operatorname{Cost}(\operatorname{MAE})}{\operatorname{Cost}(\operatorname{OPT}_{\operatorname{dynamic}})} \ge \frac{l \cdot r \cdot k^3}{4l \cdot k^2 + O(k \cdot n)} \to \Omega(k) \,.$$

The result follows for $l \to \infty$ and $k = \Omega(\sqrt{n})$.